523

NPS-MA-93-012

NAVAL POSTGRADUATE SCHOOL

Monterey, California

NONLINEAR MODE COUPLING IN FREE ELECTRON LASERS

by

C. L. Frenzen

Technical Report For Period
January 1993 - March 1993

Approved for public release; distribution unlimited

: Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93943

Rear Admiral T.A. Mercer
Superintendent

Harrison Shull
Provost

This report was prepared in conjunction with research conducted for the Naval Postgraduate School and funded by the Naval Postgraduate School.

This report was prepared by:

Nonlinear Mode Coupling in Free Electron Lasers

C.L. Frenzen
Department of Mathematics
Naval Postgraduate School
Monterey CA, 93943

Abstract

The optical field in free electron lasers can sometimes develop sidebands and exhibit very complex behavior. The purpose of this report is to develop a two mode model of the optical field in a free electron laser in which the optical modes are coupled by the free electrons in the laser. This model may be used to study the long term stability of the optical wave and the development of sidebands in the optical field. In this report the equations of motion governing the interaction of an electron with the optical modes will be derived.

1. Introduction

Initial theoretical studies of the free electron laser (FEL) used quantum mechanics ([1]–[3]). However it was shown in [4] that a classical treatment of FELs would adequately describe most phases of operation ([5]–[10]) and result in substantial simplification of the resulting equations describing their operation.

One way to understand some of the main features of the FEL problem is to consider the motion of a pendulum with weak damping and a nearly constant tangential forcing. In a mathematical model of the FEL with a tapered magnetic wiggler, however, the weak damping term in the pendulum equation cannot be ignored because the wiggler is extremely long compared to its period, and one must account for the "cumulative" effects of this damping over the entire length of the wiggler in order to obtain accurate solutions. The only way to systematically account for these higher order effects is to formulate a completely dimensionless problem which contains dimensionless parameters whose magnitudes characterize various distinguished limits.

In a FEL frequently the optical field can develop sidebands and become highly complex in its behavior. The purpose of this report is to develop a model in which there are only two modes present in the optical field: these modes are coupled to one another through the electrons in the FEL. This model can be used to study the long term stability of the optical wave and the development of sidebands in the optical field. A regime of particular interest is that of sustained resonance in the two mode model. In this report we shall derive the equations of motion governing the interaction of an electron with the optical fields in the two mode model of a FEL.

2.1 Equations of Motion

We shall formulate the equations of motion as a first order system of ordinary differential equations. Our derivation follows that in [11]. However we shall be extending the derivation given there for a single optical mode to the two optical mode case. The convention we shall use is that upper-case letters will represent dimensional variables and lower-case letters will represent dimensionless variables.

To determine the leading order equations, consider a single electron moving in a helical wiggler magnetic field given by

$$\mathbf{B}_w = B_w \left\{ \hat{\mathbf{X}} \cos \frac{2\pi}{\lambda_w} Z + \hat{\mathbf{Y}} \sin \frac{2\pi}{\lambda_w} Z \right\}, \tag{1}$$

where B_w is the magnetic field amplitude and λ_w is its wavelength. The vectors $\hat{\mathbf{X}}, \hat{\mathbf{Y}}$, and $\hat{\mathbf{Z}}$ are orthonormal and are associated with the coordinates X, Y, and Z, where Z is the coordinate along the direction of the magnet axis. We shall assume that the radiation (or signal) field is a linear superposition of two modes and is given by

$$\mathbf{E}_s = \mathbf{E}_s^{(1)} + \mathbf{E}_s^{(2)}, \tag{2}$$

where

$$\mathbf{E}_s^{(i)} = -E_s^{(i)}(Z) \left\{ \hat{\mathbf{X}} \sin \Theta_i + \hat{\mathbf{Y}} \cos \Theta_i \right\}, i = 1, 2 \tag{3}$$

and

$$\mathbf{B}_s = \mathbf{B}_s^{(1)} + \mathbf{B}_s^{(2)}, \tag{4}$$

where

$$\mathbf{B}_s^{(i)} = \hat{\mathbf{Z}} \times \mathbf{E}_s^{(i)}, i = 1, 2. \tag{5}$$

$E_s^{(i)}$ is the radiation amplitude of mode i and

$$\Theta_i = \frac{2\pi}{\lambda_s^{(i)}} Z - \omega_s^{(i)} T + \phi_s^{(i)}, i = 1, 2 \tag{6}$$

where $\lambda_s^{(i)}$ is the wavelength, $\omega_s^{(i)}$ is the frequency, T is time, and $\phi_s^{(i)}$ is the phase of mode i, i = 1, 2.

We shall assume that the electron is relativistic, though its transverse velocity (the velocity orthogonal to the Z axis) is small. Let $\boldsymbol{\beta}$ be the dimensionless electron velocity ($\boldsymbol{\beta}$ is the electron velocity divided by c, the speed of light). We shall decompose $\boldsymbol{\beta}$ into a component parallel and a component perpendicular to the Z axis:

$$\boldsymbol{\beta} = \beta_{\|} \hat{\mathbf{Z}} + \boldsymbol{\beta}_{\perp}, \qquad 0 < \beta_{\perp} \ll \beta_{\|} < 1. \tag{7}$$

The equation of motion (Newton's second law) for the electron is

$$\dot{\mathbf{P}} = e\mathbf{E}_s + e\boldsymbol{\beta} \times (\mathbf{B}_s + \mathbf{B}_w) + \text{space charge force} + \text{radiation reaction force}, \tag{8}$$

where e is the electron charge and $\mathbf{P} = \gamma \mathrm{mc} \boldsymbol{\beta}$ is the electron momentum. γ is the Lorentz factor, defined by

$$\gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}}}, \tag{9}$$

c is the speed of light, and m is the electron rest mass. Criteria for neglecting the effects of the radiation reaction force and the space charge force were given in [12], and were derived by examining higher order terms in the equations of motion. These conditions will be given later in this report.

The conservation of energy implies $(\dot{\gamma mc^2}) = ec\boldsymbol{\beta} \cdot \mathbf{E}_s$ or

$$\dot{\gamma} = \frac{\mathrm{e}}{\mathrm{mc}} \boldsymbol{\beta} \cdot \mathbf{E}_s \tag{10}$$

To gain a rough idea of the approach to be followed, we shall assume initially that only a single mode in the radiation field is present. Later we shall repeat the same calculations in more detail with two modes in the radiation field.

The momentum equation (8) implies

$$(\gamma \dot{\mathrm{mc}\boldsymbol{\beta}}) = \dot{\gamma}(\mathrm{mc}\boldsymbol{\beta}) + \gamma \mathrm{mc}\dot{\boldsymbol{\beta}} = \gamma^3(\mathrm{mc}\boldsymbol{\beta})(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}) + \gamma \mathrm{mc}\dot{\boldsymbol{\beta}}\,, \qquad (11)$$

since $\dot{\gamma} = \gamma^3 \boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}$.

Combining the momentum equation (11) and the energy equation (10), we conclude that

$$\gamma \mathrm{mc}\dot{\boldsymbol{\beta}} = e\mathbf{E}_s - e\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}_s) + e\boldsymbol{\beta} \times (\hat{\mathbf{Z}} \times \mathbf{E}_s) + e\boldsymbol{\beta} \times \mathbf{B}_w. \qquad (12)$$

Since $\boldsymbol{\beta} \times (\hat{\mathbf{Z}} \times \mathbf{E}_s) = \hat{\mathbf{Z}}(\boldsymbol{\beta} \cdot \mathbf{E}_s) - \mathbf{E}_s(\boldsymbol{\beta} \cdot \hat{\mathbf{Z}})$, equation (12) may be written as

$$\gamma \mathrm{mc}\dot{\boldsymbol{\beta}} = e(1-\beta_{||})\mathbf{E}_s + e\hat{\mathbf{Z}}\boldsymbol{\beta} \cdot \mathbf{E}_s + e\boldsymbol{\beta} \times \mathbf{B}_w. \qquad (13)$$

Using the fact that $1 - \beta_{||} \ll 1$ and that

$$\boldsymbol{\beta} \cdot \mathbf{E}_s = \boldsymbol{\beta}_{\perp} \cdot \mathbf{E}_s \quad , \beta_{\perp} \ll 1, \qquad (14)$$

we obtain the approximate equation

$$\dot{\boldsymbol{\beta}} = \frac{\mathrm{e}}{\gamma \mathrm{mc}} \boldsymbol{\beta} \times \mathbf{B}_w. \qquad (15)$$

Substituting $\boldsymbol{\beta}$, the solution of equation (15), into the energy equation (10), we find

$$\dot{\gamma} = \frac{\mathrm{eE_s}\beta_{\perp}}{\mathrm{mc}} \sin\phi, \qquad (16)$$

$$\dot{\phi} = \frac{2\pi c}{\lambda_w}\left(1 - \frac{\lambda_w}{\lambda_s}\frac{1+b_w^2}{2\gamma^2}\right), \qquad (17)$$

where $\phi = (2\pi/\lambda_w + 2\pi/\lambda_s)Z - \omega_s T + \phi_s$, and

$$b_w = \frac{eB_w\lambda_w}{2\pi mc^2}, \qquad (18)$$

is defined as the dimensionless amplitude of the wiggler in Section (2.2). The details of the above calculations (for the case of two modes) will be given in Section (2.3).

We can now define a (dimensionless) resonant electron energy γ_r for which $\dot{\phi} = 0$. This energy is obtained from equation (17) and has the value

$$\gamma_r^2 = \frac{\lambda_w}{\lambda_s}\frac{1 + b_w^2}{2}. \tag{19}$$

A free electron laser usually operates in a state for which $\gamma \approx \gamma_r$ (see [2], [6], and [7]). Equation (16) indicates that electrons with (dimensionless) energy γ_r in a magnetic wiggler with wavelength λ_w and field amplitude b_w will spontaneously emit radiation (if $\dot{\gamma}_r < 0$) or absorb radiation (if $\dot{\gamma}_r > 0$) of wavelength

$$\lambda_s = \frac{\lambda_w(1 + b_w^2)}{2\gamma_r^2}. \tag{20}$$

2.2 Normalization and Nondimensionalization

It is very important to decide on characteristic scales for the dependent and independent variables in the study of a system of differential equations by asymptotic methods. The appropriate choice of scales is usually dictated by the physical regime of interest represented in the system under study and can, in fact, be changed to suit a different regime. They key point is that once a set of scales is chosen all dependent and independent variables become *dimensionless and order one in magnitude.* There then is no ambiguity about the size or numerical magnitude of the dependent and independent variables. Of course, there are now dimensionless parameters present in the problem which are functions of the characteristic scales. It is *their* orders of magnitude which indicate what physical regime is under study, and what is small, etc.

We shall use $\lambda_w/2\pi = L_0$ as a characteristic unit of length and $L_0/c = T_0$ as a characteristic unit of time. Let $2\pi/\lambda_w = K_w$ be the wavenumber of the wiggler. Then in terms of these units we define the dimensionless variables

$$z = K_w Z = \frac{Z}{L_0} \quad , \quad t = K_w c T = \frac{T}{T_0}. \tag{21}$$

(Note that lower case variables will always be dimensionless.)

We now introduce the dimensionless field variables:

$$b_w = \frac{e}{K_w mc^2}B_w, \quad b_s = \frac{e}{K_w mc^2}B_s, \quad e_s = \frac{e}{K_w mc^2}E_s, \tag{22}$$

where $mc^2 = 0.511$ MeV. Using the normalizations introduced in equations (21) and (22), equations (8) and (10) may be written in dimensionless form as

$$\frac{d(\gamma\boldsymbol{\beta})}{dt} = \mathbf{e}_s + \boldsymbol{\beta} \times (\mathbf{b}_s + \mathbf{b}_w) + \text{higher order terms}, \tag{23}$$

$$\frac{d\gamma}{dt} = \boldsymbol{\beta} \cdot \mathbf{e}_s. \tag{24}$$

In equation (23), the higher order terms include the normalized space charge force and the normalized radiation force.

To characterize the order of magnitude of each term in these equations we need to introduce a small parameter $\varepsilon, 0 < \varepsilon \ll 1$. One way to do this is to choose the parameter ε to be a measure of the slow variation of the wiggler field along the magnet axis. To do this we define ε as

$$\varepsilon = \frac{\lambda_w}{2\pi L} = \frac{L_0}{L}, \quad \text{or} \quad \varepsilon = \frac{1}{l} \tag{25}$$

where L and l are respectively the dimensional and dimensionless length of the wiggler and L_0 is the characteristic length defined by the wiggler in equation (21). The smallness of ε signifies that the magnetic wiggler produces a small effect which, for an electron moving along the magnet axis, lasts for a long time.

Using the small parameter ε we can define a "slow" variable $\tilde{z}$ as

$$\tilde{z} = \varepsilon z \ , \quad 0 < \varepsilon \ll 1. \tag{26}$$

The slow variable $\tilde{z}$ enables us to define "slowly varying" functions of z in the following sense: an O(1) function of z which varies slowly along the wiggler axis while maintaining its order of magnitude can be written as $f(\tilde{z})$. Note that df/dz = $O(\varepsilon)$. A slowly varying function f of the dimensionless variable z may be written as

$$f(\tilde{z}) = f(\varepsilon z) = f(\frac{L_0}{L}\frac{Z}{L_0}) = f(\frac{Z}{L}),$$

so that slowly varying means changes occur over length scales which are much smaller than the length of the wiggler.

Typically the wiggler length L is several meters and the wavelength λ_w of the wiggler's magnetic field is on the order of a centimeter, so ε is a small number, about 10^{-3}. Also since the dimensionless wiggler length l is $1/\varepsilon$, the wiggler must be considered long in the mathematical sense. This means that the cumulative effects of higher-order terms may be important.

We will now denote all normalized (dimensionless) variables with an overbar. The wiggler magnetic field is then written in normalized form as

$$\mathbf{b}_w = \bar{b}_w(\tilde{z})\{\hat{\mathbf{x}} \cos z + \hat{\mathbf{y}} \sin z\}. \tag{27}$$

It can be seen now from equation (19) that $b_w \gg 1$ leads to $\gamma_r^2 \gg \frac{\lambda_w}{\lambda_s}$, ie., extra high energy electrons are required. On the other hand, if $b_w \ll 1$, the wiggler field is then too weak to significantly decelerate electrons. We conclude that the only reasonable choice is $b_w = \bar{b}_w(\tilde{z}) = O(1)$. Because the operation of free electron lasers is near the resonant state defined in equation (19) (see references [2], [6] , [7]), and since typical values of γ_r are large, we shall write the dimensionless energy of an electron and the dimensionless wavenumber of the signal in the following form consistent with equation (19):

$$\gamma = \varepsilon^{-\alpha/2}\bar{\gamma}, \tag{28}$$

$$\frac{\lambda_w}{\lambda_s} = \varepsilon^{-\alpha}\bar{k}_s. \tag{29}$$

In the above equation, $\bar{k}_s$ is the O(1) dimensionless signal wavenumber and $\varepsilon^\alpha \equiv \mu$ is a second small parameter which measures the ratio λ_s/λ_w. The determination of μ comes from a consideration of the physical regime under discussion. Recall that variables with an overbar are dimensionless and O(1).

After this preliminary analysis, let us assume again that there are two modes present in the radiation field. In normalized form its components appear as

$$\mathbf{e}_s^{(i)} = -\varepsilon^\xi \bar{\mathbf{e}}_s^{(i)} = \{\hat{\mathbf{x}} \sin \bar{\theta}_i + \hat{\mathbf{y}} \cos \bar{\theta}_i\}, \tag{30}$$

$$\mathbf{b}_s^{(i)} = \hat{\mathbf{z}} \times \mathbf{e}_s^{(i)}, \tag{31}$$

for i= 1, 2, and where

$$\bar{\theta}_i = \varepsilon^{-\alpha}\bar{k}_s^{(i)}(z-t) + \phi_s^{(i)}, \tag{32}$$

again for i = 1,2. Note that the total radiation field is given, as before, by equations (2) and (4). The parameter ξ in equation (30) is again determined by the physical regime under discussion. We shall show in section [2.3] that the only really interesting limit for the free electron laser is $\xi = 0$.

The radiation parameters $\bar{\mathbf{e}}_s^{(i)}$ and $\phi_s^{(i)}$ depend on the wave evolution process and must be determined by Maxwell's equations. There are some open questions here which will be discussed later. We will see that $\bar{\mathbf{e}}_s$ evolves on a slower scale than $\bar{b}_w$.

There are basically three types of magnetic wigglers or undulators. There are those in which λ_w varies, those in which b_w varies, and those in which both λ_w and b_w vary. Our approach to this problem can be adapted to any of these wigglers. However, the work in [7] suggests that the constant λ_w case is probably the best method for enhancing efficiency. Since it also offers apparent hardware advantages we will restrict our attention to this case.

2.3 Higher Order Correction Terms

In this section we calculate higher order correction terms for the equations of motion and we mention conditions derived in [12] for neglecting the space-charge force and the radiation reaction force.

We begin by looking at equation governing the temporal evolution of the perpendicular component of the velocity, $\vec{\beta_\perp}$. From equations (23) and (24) we find

$$\epsilon^{-\alpha/2}\frac{d}{dt}(\vec{\beta_\perp}) = \mathbf{e_s} + (\beta_{||}\hat{z} + \vec{\beta_\perp}) \times (\hat{z} \times \mathbf{e_s} + \mathbf{b_w}). \tag{33}$$

Since $\mathbf{e_s}$ is perpendicular to $\hat{z}$, we find that the last equation may be written

$$\epsilon^{-\alpha/2}\frac{d}{dt}(\vec{\beta_\perp}) = (1-\beta_{||})\mathbf{e_s} + \beta_{||}\hat{z} \times \mathbf{b_w} + (\text{ higher order terms}) \tag{34}$$

Now , for a single electron, $\frac{dz}{dt} = \beta_{||}$. We can regard this as an ordinary differential equation expressing z as a function of t. A little later it will be convenient to invert this relationship and to express t as a function of z.

From [12], the following conditions must hold in order to neglect the radiation force:

$$(\frac{10^{-13}cm}{\lambda_s}) = o(\epsilon^{\alpha+\xi}) \tag{35}$$

Conditions necesssary for neglecting the space charge force are

$$\frac{\omega_p^2}{\omega_s^2} = o(\epsilon^{5\alpha/2+\xi}) \tag{36}$$

Once α is determined (and it will be shown later that ξ is zero), the above two equations provide conditions which must hold in order to consistently neglect the effects of the radiation and space charge forces on an electron. In the second equation, the ratio $\frac{\omega_p^2}{\omega_s^2}$ is the ratio of the plasma frequency to the signal frequency. It will turn out that this ratio is a fundamental parameter measuring the rate of evolution of the radiation field. Consequently the second condition on the neglect of the radiation force plays an important role in the evolution of the optical field.

Inserting the specific field quantities into equation (34) then gives

$$\begin{aligned}
\epsilon^{-\alpha/2}\frac{d}{dt}(\vec{\beta_\perp}) &= -\beta_{||}\bar{b}_w(\tilde{z})\,(\hat{x}\sin z - \hat{y}\cos z) \\
&\quad -(1-\beta_{||})\epsilon^{\xi}\bar{b}_s^{(1)}\left(\hat{x}\sin\bar{\theta}_1 + \hat{y}\cos\bar{\theta}_1\right) \\
&\quad -(1-\beta_{||})\epsilon^{\xi}\bar{b}_s^{(2)}\left(\hat{x}\sin\bar{\theta}_2 + \hat{y}\cos\bar{\theta}_2\right) \\
&\quad +\text{higher order terms},
\end{aligned} \tag{37}$$

where $\bar{b}_s^i = \bar{e}_s^i$ since $\mathbf{b}_\mathrm{s}^{(\mathrm{i})} = \hat{z}\times\mathbf{e}_\mathrm{s}^{(\mathrm{i})}$.

Noting that $\beta_{||}\,dt = dz$, we integrate equation (37) to obtain the form

$$\begin{aligned}
\vec{\beta_\perp} &= \epsilon^{\alpha/2}\frac{\bar{b}_w}{\bar{\gamma}}\,(\hat{x}\cos z + \hat{y}\sin z) \\
&\quad -\epsilon^{3\alpha/2}\frac{\bar{b}_w'}{\bar{\gamma}}\,(\hat{x}\sin z - \hat{y}\cos z) \\
&\quad -\epsilon^{3\alpha/2+\xi}\frac{\bar{b}_s^{(1)}}{\bar{\gamma}\bar{k}_s^{(1)}}\left(\hat{x}\cos\bar{\theta}_1 - \hat{y}\sin\bar{\theta}_1\right) \quad . \\
&\quad -\epsilon^{3\alpha/2+\xi}\frac{\bar{b}_s^{(2)}}{\bar{\gamma}\bar{k}_s^{(2)}}\left(\hat{x}\cos\bar{\theta}_2 - \hat{y}\sin\bar{\theta}_2\right) \\
&\quad +O(\epsilon^{5\alpha/2})
\end{aligned} \tag{38}$$

where $\bar{\theta}_i = \epsilon^{-\alpha}\bar{k}_s^{(i)}(z-t) + \phi_s^{(i)}$. In equation (38), $\bar{b}_w'$ stands for the derivative of $\bar{b}_w$ with respect to $\epsilon^\alpha z$. However since $\bar{b}_w$ is a function of $\tilde{z} = \epsilon z$, it follows that

$$\frac{d\overline{b}_w}{d(\epsilon^\alpha z)} = O(\epsilon^{1-\alpha}) \tag{39}$$

The $O(\epsilon^{5\alpha/2})$ error in equation (38) is due to ignoring the variations of $d\overline{b}_w/d(\epsilon^{\alpha z}), \overline{b}_s$, and ϕ_s with $\tilde{z}$. To obtain (38) , we have also assumed that $\beta_\perp(0^-) = O(\epsilon^{5\alpha/2})$, ie., the transverse velocity of the incoming beam can be ignored before it enters the magnetic wiggler.

Now

$$\frac{d\overline{\gamma}}{dt} = \epsilon^{\alpha/2}\beta \cdot \mathbf{e}_\mathrm{s} = \beta_\perp \cdot (\mathbf{e}_\mathrm{s}^{(1)} + \mathbf{e}_\mathrm{s}^{(2)}) \tag{40}$$

since the component of the electron velocity parallel to the wiggler axis is orthogonal to the signal fields. Inserting equations (37) and (38) into the above expression in (40) , we find after some calculation and rearrangement that

$$\begin{aligned}\frac{d\overline{\gamma}}{dt} &= -\epsilon^{\alpha+\xi}\left[\frac{\overline{b}_w\overline{b}_s^{(1)}}{\overline{\gamma}}\sin(z+\overline{\theta_1}) + \frac{\overline{b}_w\overline{b}_s^{(2)}}{\overline{\gamma}}\sin(z+\overline{\theta_2})\right] \\ &\quad -\epsilon^{2\alpha+\xi}\left[\frac{\overline{b}'_w\overline{b}_s^{(1)}}{\overline{\gamma}}\cos(z+\overline{\theta_1}) + \frac{\overline{b}'_w\overline{b}_s^{(2)}}{\overline{\gamma}}\cos(z+\overline{\theta_2})\right] \\ &\quad +\epsilon^{2\alpha+2\xi}\left[\frac{\overline{b}_s^1\overline{b}_s^{(2)}}{\overline{\gamma}\overline{k}_s^{(1)}}\sin(\overline{\theta_2}-\overline{\theta_1}) + \frac{\overline{b}_s^1\overline{b}_s^{(2)}}{\overline{\gamma}\overline{k}_s^{(2)}}\sin(\overline{\theta_1}-\overline{\theta_2})\right] \\ &\quad +O(\epsilon^{3\alpha}).\end{aligned} \tag{41}$$

We shall now define the phases $\phi_i, i = 1, 2$ by

$$\phi_i = z + \overline{\theta_i} = z + \epsilon^{-\alpha}\overline{k}_s^{(i)}(z-t) + \phi_s^{(i)}. \tag{42}$$

We observe that since many of the parameters in the problem depend on z, as previously mentioned, it will be convenient to use d/dz rather than d/dt for the equations of motion. The velocity dz/dt can be determined from the relationships dz/dt = $\beta_{||}$ and

$$\beta_{||} = \left(1 - 1/\gamma^2 - \beta_\perp^2\right)^{1/2}. \tag{43}$$

Using the above equation and the expressions for γ and $\beta_\perp$ in equations (9) and (38) , we find after some effort, that

$$
\begin{aligned}
\frac{1}{\beta_{\|}} &= 1+\epsilon^{\alpha}\left(\frac{1+\bar{b}_w^2}{2\bar{\gamma}^2}\right)+\epsilon^{2\alpha}3/2\left(\frac{1+\bar{b}_w^2}{2\bar{\gamma}^2}\right)^2 \\
&-\epsilon^{2\alpha+\xi}\left[\frac{\bar{b}_w\bar{b}_s^{(1)}}{\bar{\gamma}^2\bar{k}_s^{(1)}}\cos\phi_1+\frac{\bar{b}_w\bar{b}_s^{(2)}}{\bar{\gamma}^2\bar{k}_s^{(2)}}\cos\phi_2\right] \\
&+O(\epsilon^{3\alpha})
\end{aligned} \tag{44}
$$

Since

$$\frac{d\bar{\gamma}}{dt}=\frac{dz}{dt}\frac{d\bar{\gamma}}{dz} \tag{45}$$

it follows that

$$\frac{d\bar{\gamma}}{dz}=\frac{1}{\beta_{\|}}\frac{d\bar{\gamma}}{dt}. \tag{46}$$

We thus can combine equations (41) and (44) above to derive an expression for the rate of change of $\bar{\gamma}$ with respect to the dimensionless distance along the wiggler axis, z. We find after some calculation that

$$
\begin{aligned}
\frac{d\bar{\gamma}}{dz} &= -\epsilon^{\alpha+\xi}\left(\frac{\bar{b}_w\bar{b}_s^{(1)}}{\bar{\gamma}}\sin\phi_1+\frac{\bar{b}_w\bar{b}_s^{(2)}}{\bar{\gamma}}\sin\phi_2\right) \\
&-\epsilon^{2\alpha+\xi}\left[\frac{1+\bar{b}_w^2}{2\bar{\gamma}^2}\right]\left[\frac{\bar{b}_w\bar{b}_s^{(1)}}{\bar{\gamma}}\sin\phi_1+\frac{\bar{b}_w\bar{b}_s^{(2)}}{\bar{\gamma}}\sin\phi_2\right] \\
&-\epsilon^{2\alpha+\xi}\left[\frac{\bar{b}_w'}{\bar{\gamma}}\left(\bar{b}_s^{(1)}\cos\phi_1+\bar{b}_s^{(2)}\cos\phi_2\right)\right] \\
&+\epsilon^{2\alpha+2\xi}\left[\frac{\bar{b}_s^{(1)}\bar{b}_s^{(2)}}{\bar{\gamma}}\sin(\phi_2-\phi_1)\left(\frac{1}{\bar{k}_s^{(1)}}-\frac{1}{\bar{k}_s^{(2)}}\right)\right] \\
&+O(\epsilon^{3\alpha+\xi}).
\end{aligned} \tag{47}
$$

Equation (47) determines the evolution of $\bar{\gamma}$ with respect to z, and it can be seen that this depends on the field quantities, *and* on the phases ϕ_1, ϕ_2 defined in (42). We now determine the evolution of these latter quantities with respect to z. Differentiating the expression for ϕ_1 in equation (42) with respect to z, recalling that $\frac{dt}{dz} = 1/\beta_{||}$, and using equation (44) for $1/\beta_{||}$, we find that

$$\begin{aligned}\frac{d\phi_1}{dz} &= 1 - \bar{k}_s^{(1)}\left(\frac{1+\bar{b}_w^2}{2\bar{\gamma}^2}\right) + \epsilon^\alpha\left[-3/8\bar{k}_s^{(1)}\left(\frac{1+\bar{b}_w^2}{\bar{\gamma}^2}\right)^2 + \phi_s^{(1)\prime}\right] \\ &+\epsilon^{\alpha+\xi}\bar{k}_s^{(1)}\left[\frac{\bar{b}_w\bar{b}_s^{(1)}}{\bar{\gamma}^2\bar{k}_s^{(1)}}\cos\phi_1 + \frac{\bar{b}_w\bar{b}_s^{(2)}}{\bar{\gamma}^2\bar{k}_s^{(2)}}\cos\phi_2\right] + \epsilon\phi_s^{(1)\prime} \\ &+O(\epsilon^{2\alpha})\end{aligned} \tag{48}$$

We can obtain an expression for the rate of change of ϕ_2 with respect to z in a similar fashion:

$$\begin{aligned}\frac{d\phi_2}{dz} &= 1 - \bar{k}_s^{(2)}\left(\frac{1+\bar{b}_w^2}{2\bar{\gamma}^2}\right) + \epsilon^\alpha\left[-3/8\bar{k}_s^{(2)}\left(\frac{1+\bar{b}_w^2}{\bar{\gamma}^2}\right)^2 + \phi_s^{(2)\prime}\right] \\ &+\epsilon^{\alpha+\xi}\bar{k}_s^{(2)}\left[\frac{\bar{b}_w\bar{b}_s^{(1)}}{\bar{\gamma}^2\bar{k}_s^{(1)}}\cos\phi_1 + \frac{\bar{b}_w\bar{b}_s^{(2)}}{\bar{\gamma}^2\bar{k}_s^{(2)}}\cos\phi_2\right] + \phi_s^{(2)\prime} \\ &+O(\epsilon^{2\alpha}),\end{aligned} \tag{49}$$

where $\phi_s^{(i)\prime} = d\phi_s^{(i)}/dz$ must be determined by using Maxwell's equations to determine the evolution of the optical field.

Note that the electron energy ($\bar{\gamma}$ in (47) depends on both the phases ϕ_1, ϕ_2, whose evolution is determined by equations (48) and (49). Note also that the evolution of these phases depends on the electron energy $\bar{\gamma}$.

If we now compare the higher order terms in the expressions for d$\bar{\gamma}$/dz, dϕ_1/dz, and dϕ_2/dz, we see that unless $\xi = 0$, oscillatory solutions for $\bar{\gamma}, \phi_1$, and ϕ_2 will not exist. Since the existence of oscillations in $\bar{\gamma}$ implies a transfer of energy from the electron to the optical field, and this is essential to the operation of the FEL, we find that we must take $\xi = 0$ in equations (47), (48), and (49).

At this point we have two *independent* small parameters: ϵ and $\epsilon^\alpha \equiv \mu$ (as we have not yet specified α). From equation (25) ϵ is a measure of the smallness of λ_w compared to L, while $\mu^{-1/2}$ measures the energy γ of the electron (see (28)). As in [12], near the resonant state ($\gamma \approx \gamma_r$), there are two distinguished limits corresponding to $\alpha = 0$ and $\alpha = 2/3$. The latter case is known as "slow resonance" and is substantially more difficult to solve than the case when $\alpha = 1$, and for this reason we shall restrict our attention to the $\alpha = 1$ case.

Equations (47), (48), and (49) form three coupled ordinary differntial equations for $\bar{\gamma}, \phi_1$, and ϕ_2. The coefficients in these ODEs are slowly varying functions of $\tilde{z} = \epsilon z$. Though the single mode case (see [12]) can be cast into Hamiltonian form, it does not seem possible to express the two mode equations in Hamiltonian form. One must also study how the optical wave interacts with and evolves according to the presence of the electron beam. (We have derived the equations for a *single* electron, but generally in the electron beam of a FEL there are many electrons).

In order to maintain the coherence and monochromicity of the radiation, the evolution of the optical wave should be slow. Thus the leading order parts of the signal parameters $b_s^{(i)}$ and ϕ_s^i must either be constant or depend on the prescribed slowly varying wiggler parameters. These signal parameters in the radiation field are determined by the wave evolution process which is governed by Maxwell's equations. Though determination of the asymptotic behavior of the signal parameters is relatively straight forward in the one mode model (see [12] for example), it still is not quite clear how the wave evolution occurs in the two mode model.

We leave the further details of this analysis, and the study of the three ODEs in (47), (48), and (49) to a future report.

REFERENCES

[1.] J.M.J. Madey, "Stimulated emission of bremsstrahlung in a periodic magnetic field," *J. Appl. Phys.*, vol. 42, pp. 1906–1913, 1971.

[2.] V.P. Sukhatme and P. W. Wolf, "Stimulated Compto scattering as a radiation source–theoretical limitations," *J. Appl. Phys.*, vol.44, pp. 2331–2334, 1973.

[3.] J.M.J. Madey, H.A. Schwettman, and W.M. Fairbank, "A free electron laser," *IEEE Trans. Nucl. Sci.*, vol. NS–20, pp. 980–983, June 1973.

[4.] F.A. Hopf, P. Meystre. M.O. Scully and W.H. Louisell, "Classical theory of a free electron laser," *Phys. Rev. Lett.,* vol. 37, pp. 1215–1218, 1976.

[5.] W.B. Colson, "Free electron laser wave and particle dynamics," in *Free Electron Lasers*, S. Martellucci and A.N. Chester, Eds. New York: Plenum, 1980, pp. 189–209.

[6.] C. Pellegrini,"Physics of the free electron laser," in *Free Electron Lasers*, S. Martellucci and A.N. Chester, Eds. New York: Plenum, 1980, pp. 91–119.

[7.] N.M. Kroll, P.L. Morton and M.N. Rosenbluth, "Free electron lasers with variable wigglers," *IEEE J. Quantum Electron.*, vol. QE–17, pp. 1436–1468, 1981.

[8.] J.C. Garrison and J. Wong, "Correction to the adiabatic approximation for variable parameter free electron lasers," *IEEE J. Quantum Electron.*, vol. QE–17, pp. 1469–1475, 1981.

[9.] P. Sparangle, C.M. Tang and I. Bernstein, "Evolution of spontaneous and coherent radiation in the free electron laser oscillator," *Phys. Rev. A,* vol. 28, pp. 2300–2309, 1983.

[10.] T.C. Marshall, *Free Electron Lasers.* New York: Macmillan, 1985.

[11.] Y.P. Li and J. Kevorkian, "Explicit approximations for strictly nonlinear oscillators with slowly varying parameters with applications to free electron lasers," *Stud. Appl. Math.,* vol. 78, no. 2 , 1988.

[12.] Y.P. Li, "Free electron lasers with variable wiggle parameters; a strictly nonlinear oscillator with slowly varying parameters," Ph.D. dissertation, Univ. Washington, Seattle, June 1987.

DISTRIBUTION LIST

Director Defense Tech Information Center Cameron Station Alexandria, VA 22314	(2)
Research Office Code 81 Naval Postgraduate School Monterey, CA 93943	(1)
Library Code 52 Naval Postgraduate School Monterey, CA 93943	(2)
Professor Richard Franke Department of Mathematics Naval Postgraduate School Monterey, CA 93943	(1)
Dr. Neil L. Gerr Mathematical Sciences Division Office of Naval Research 800 North Quincy Street Arlington, VA 22217-5000	(1)
Dr. Richard Lau Mathematical Sciences Division Office of Naval Research 800 North Quincy Street Arlington, VA 22217-5000	(1)
W.B. Colson Department of Physics Naval Postgraduate School Monterey, CA 93943	(1)
V. Smiley, Program Manager NOSC Code 808 271 Catalina Blvd. San Diego, CA 92129	(1)
C.L. Frenzen Department of Mathematics Naval Postgraduate School Monterey, CA 93943	(10)